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2003 J. Phys. A: Math. Gen. 36 6875

(http://iopscience.iop.org/0305-4470/36/24/321)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) 6875–6893 PII: S0305-4470(03)60590-1

Free-field realization of boundary states and boundary correlation functions of minimal models

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Received 10 March 2003, in final form 24 April 2003
Published 5 June 2003
Online at stacks.iop.org/JPhysA/36/6875

Abstract
We propose a general formalism to compute exact correlation functions for Cardy's boundary states. Using the free-field construction of boundary states and applying the Coulomb-gas technique, it is shown that charge neutrality conditions pick up particular linear combinations of conformal blocks. As an example we study the critical Ising model with free and fixed boundary conditions, and demonstrate that conventional results are reproduced. This formalism thus directly associates algebraically constructed boundary states with correlation functions which are in principle observable or numerically calculable.

PACS numbers: 11.10.Kk, 11.25.Hf, 68.35.Rh

1. Introduction

Modular invariance of partition functions plays extremely important roles in two-dimensional conformal field theory (CFT). The ADE classification of modular invariants by Cappelli *et al* [1, 2] (see also [3–5]) is obtained by considering CFTs on the torus. The classified modular invariants correspond to particular sets of operators, which are supposed to model critical systems in certain universality classes. Similar consideration also applies to CFTs defined on a manifold with boundary. For a CFT on the cylinder, the constraints from modular invariance lead to a classification of boundary states. This method was invented by Cardy [6] in the eighties and developed by many in the nineties [7–12]. Such a classification of boundary states with consistent modular properties has recently attracted much attention along with the development of D-brane/string theory and various applications of boundary CFT to statistical physics.

Boundary states with consistent modular properties, or *consistent* boundary states, are normally believed to represent boundary conditions which may be physically imposed on D-branes or borders of statistical systems. In order to understand the behaviour of observables in the presence of such a boundary, we need to find correlation functions for consistent boundary states which are defined through Cardy's classification. In principle this can be accomplished by operator product expansions (OPEs), that is, by finding boundary operators for a given boundary state, solving the constraints satisfied by coupling constants and then obtaining the correlation functions by OPEs using the boundary operators. The correlation functions obtained in this way are perturbative, that is, in the form of series expansion. For practical use, we often need to know exact correlation functions and in such a case we normally solve differential equations to find conformal blocks and fix their coefficients by physical considerations [13]. In this differential equation method, however, the relation to Cardy's classification of boundary states is not quite evident.

In this paper, we present an alternative formalism for finding boundary correlation functions[1] directly using boundary states obtained by Cardy's method. It is well known in string theory that correlation functions are simply given by inserting vertex operators within amplitudes (with or without boundaries). This picture is generalized to non-bosonic ($c \neq 1$) CFTs using the Coulomb-gas formalism on arbitrary Riemann surfaces (without boundaries) [14–17]. The purpose of this paper is to present a method of calculating boundary correlation functions based on the Coulomb-gas picture, by using the free-field representation of boundary states developed in [18]. Driven by a similar motivation, Coulomb-gas system on the half-plane is discussed in [19], where the Ising model conformal blocks are reproduced using the contour integration technique of [20] and insertion of boundary operators. The key ingredient of the formalism proposed in this paper is boundary states having boundary charges, which account *both* for the contour integral expression of conformal blocks *and* for their coefficients. The role of such boundary states has not been fully investigated so far in this context, and our formalism allows a systematic study on the relation between correlation functions and algebraically defined boundary states. In the following we shall mainly consider Coulomb-gas systems on the unit disc, where Felder's charged bosonic Fock space (CBFS) construction [14] is readily used. Although our method itself is quite general, we shall focus on presenting the ideas in simplest cases and show that it reproduces known results obtained by the conventional approach.

We organize the rest of this paper as follows. In the next section we fix our notation and review the free-field construction of boundary states [18]. We describe in section 3 our method of computing correlation functions on the disc and on the half-plane. In section 4 we illustrate the method using the Ising model and show that it reproduces the results of [13]. Finally in section 5 we summarize and conclude.

2. Boundary states in Coulomb-gas formalism

Let us start, for the sake of self-containedness, by summarizing the Coulomb-gas construction of boundary states [18]. The idea is to define coherent states in the charged bosonic Fock spaces (CBFSs) and find conditions for their diffeomorphism invariance and modular consistency.

2.1. Coulomb-gas and charged bosonic Fock space

In the Coulomb-gas formalism [21], the Virasoro minimal models are realized by the action

$$\mathcal{S} = \frac{1}{8\pi} \int \mathrm{d}^2 x \sqrt{g} (\partial_\mu \Phi \partial^\mu \Phi + 2\sqrt{2}\alpha_0 \, \mathrm{i} \Phi R) \tag{1}$$

[1] We use this term for correlation functions of *bulk* operators in the presence of boundary.

where the scalar field $\Phi(x)$ is assumed to decouple into two chiral sectors, $\Phi(z,\bar{z}) = \varphi(z) + \bar{\varphi}(\bar{z})$. The energy–momentum tensor is obtained by the variation of the action (1) as

$$T(z) = -2\pi T_{zz} = -\tfrac{1}{2} : \partial\varphi\partial\varphi : +\mathrm{i}\sqrt{2}\alpha_0\partial^2\varphi. \tag{2}$$

The central charge of this system is

$$c = 1 - 24\alpha_0^2. \tag{3}$$

The vertex operators

$$V_\alpha(z) =: \mathrm{e}^{\mathrm{i}\sqrt{2}\alpha\varphi(z)} : \tag{4}$$

are chiral fields of conformal dimensions $h_\alpha = \alpha^2 - 2\alpha_0\alpha$. In particular, the primary fields $\phi_{r,s}(0 < r < p', 0 < s < p)$ of a minimal model are realized by the vertex operators $V_{\alpha_{r,s}}(z)$ with

$$\alpha_{r,s} = \tfrac{1}{2}(1-r)\alpha_+ + \tfrac{1}{2}(1-s)\alpha_- \tag{5}$$

where $\alpha_+ = \sqrt{p/p'}, \alpha_- = -\sqrt{p'/p}$, and p and p' (we assume $p > p'$) are the two coprime integers characterizing the minimal model. The conformal dimensions of the operators are

$$h_{r,s} = \tfrac{1}{4}(r\alpha_+ + s\alpha_-)^2 - \alpha_0^2 \tag{6}$$

which agree with the Kac formula.

The holomorphic chiral boson field is expanded in modes as

$$\varphi(z) = \varphi_0 - \mathrm{i}a_0 \ln z + \mathrm{i}\sum_{n\neq 0}\frac{a_n}{n}z^{-n} \tag{7}$$

and likewise for the antiholomorphic counterpart,

$$\bar{\varphi}(\bar{z}) = \bar{\varphi}_0 - \mathrm{i}\bar{a}_0 \ln \bar{z} + \mathrm{i}\sum_{n\neq 0}\frac{\bar{a}_n}{n}\bar{z}^{-n}. \tag{8}$$

As we try to consider boundary CFT in the Coulomb-gas picture, a subtlety arises in the treatment of zero-mode, since the zero-mode of $\Phi(z,\bar{z})$ does not naturally decouple into the holomorphic and antiholomorphic sectors. In our formalism, we shall simply split it into two identical and independent copies. In this sense, the boundary theory we shall consider is not exactly a non-chiral theory on a manifold with boundary but rather a chiral theory on its Schottky double[2]. We thus have two copies of Heisenberg operators, satisfying the algebra

$$\begin{aligned} &[a_m, a_n] = m\delta_{m+n,0} \qquad [\varphi_0, a_0] = \mathrm{i} \\ &[\bar{a}_m, \bar{a}_n] = m\delta_{m+n,0} \qquad [\bar{\varphi}_0, \bar{a}_0] = \mathrm{i} \end{aligned} \tag{9}$$

with no interaction,

$$[a_m, \bar{a}_n] = 0 \qquad [\varphi_0, \bar{a}_0] = [\bar{\varphi}_0, a_0] = [\varphi_0, \bar{\varphi}_0] = 0. \tag{10}$$

In terms of the Heisenberg operators, the Virasoro operators are written as

$$L_{n\neq 0} = \frac{1}{2}\sum_{k\in\mathbb{Z}} a_{n-k}a_k - \sqrt{2}\alpha_0(n+1)a_n \tag{11}$$

$$L_0 = \sum_{k\geqslant 1} a_{-k}a_k + \frac{1}{2}a_0^2 - \sqrt{2}\alpha_0 a_0 \tag{12}$$

and likewise for the antiholomorphic counterparts.

[2] The Schottky double is a Riemann surface obtained by doubling the manifold except for boundaries (see, e.g., [22]).

The Hilbert space of CFT is realized in CBFSs with BRST projection [14]. For a *chiral* theory the CBFS $F_{\alpha;\alpha_0}$ is defined as a space obtained by operating with the creation operators $a_{n<0}$ on the chiral highest weight state $|\alpha;\alpha_0\rangle$ (see [14, 18, 23]). Since the boundary intertwines the two chiral sectors, we need to construct non-chiral Fock spaces $\mathcal{F}_{\alpha,\bar{\alpha};\alpha_0}$ in order to describe boundary states. We denote the Möbius invariant non-chiral vacua with background charge α_0 as[3] $\langle 0,0;\alpha_0|$ and $|0,0;\alpha_0\rangle$, and define highest weight vectors $\langle\alpha,\bar{\alpha};\alpha_0|$ and $|\alpha,\bar{\alpha};\alpha_0\rangle$ as

$$\langle\alpha,\bar{\alpha};\alpha_0| = \langle 0,0;\alpha_0|\,\mathrm{e}^{-\mathrm{i}\sqrt{2}\alpha\varphi_0}\,\mathrm{e}^{-\mathrm{i}\sqrt{2}\bar{\alpha}\bar{\varphi}_0} \tag{13}$$

$$|\alpha,\bar{\alpha};\alpha_0\rangle = \mathrm{e}^{\mathrm{i}\sqrt{2}\alpha\varphi_0}\,\mathrm{e}^{\mathrm{i}\sqrt{2}\bar{\alpha}\bar{\varphi}_0}|0,0;\alpha_0\rangle. \tag{14}$$

The state $\langle\alpha,\bar{\alpha};\alpha_0|$ has holomorphic and antiholomorphic charges $-\alpha$ and $-\bar{\alpha}$, respectively. Likewise, $|\alpha,\bar{\alpha};\alpha_0\rangle$ has holomorphic and antiholomorphic charges α and $\bar{\alpha}$. Using the Heisenberg algebra (9) it is easy to verify that these states satisfy

$$\langle\alpha,\bar{\alpha};\alpha_0|a_0 = \langle\alpha,\bar{\alpha};\alpha_0|\sqrt{2}\alpha \tag{15}$$

$$\langle\alpha,\bar{\alpha};\alpha_0|\bar{a}_0 = \langle\alpha,\bar{\alpha};\alpha_0|\sqrt{2}\bar{\alpha} \tag{16}$$

$$a_0|\alpha,\bar{\alpha};\alpha_0\rangle = \sqrt{2}\alpha|\alpha,\bar{\alpha};\alpha_0\rangle \tag{17}$$

$$\bar{a}_0|\alpha,\bar{\alpha};\alpha_0\rangle = \sqrt{2}\bar{\alpha}|\alpha,\bar{\alpha};\alpha_0\rangle. \tag{18}$$

These highest weight vectors are eigenstates of the Virasoro zero-modes:

$$L_0|\alpha,\bar{\alpha};\alpha_0\rangle = (\alpha^2 - 2\alpha\alpha_0)|\alpha,\bar{\alpha};\alpha_0\rangle \tag{19}$$

$$\bar{L}_0|\alpha,\bar{\alpha};\alpha_0\rangle = (\bar{\alpha}^2 - 2\bar{\alpha}\alpha_0)|\alpha,\bar{\alpha};\alpha_0\rangle. \tag{20}$$

The states $|\alpha,\bar{\alpha};\alpha_0\rangle$ are annihilated by $a_{n>0}$, $\bar{a}_{n>0}$, $L_{n>0}$, $\bar{L}_{n>0}$, and $\langle\alpha,\bar{\alpha};\alpha_0|$ are annihilated by $a_{n<0}$, $\bar{a}_{n<0}$, $L_{n<0}$, $\bar{L}_{n<0}$. The non-chiral CBFS $\mathcal{F}_{\alpha,\bar{\alpha};\alpha_0}$ is built on the highest weight vector $|\alpha,\bar{\alpha};\alpha_0\rangle$ by operating with $a_{n<0}$ and $\bar{a}_{n<0}$. The dual space $\mathcal{F}^*_{\alpha,\bar{\alpha};\alpha_0}$ is defined similarly, by acting with $a_{n>0}$ and $\bar{a}_{n>0}$ on $\langle\alpha,\bar{\alpha};\alpha_0|$. The non-chiral CBFSs thus defined are essentially the direct products of chiral CBFSs, $\mathcal{F}_{\alpha,\bar{\alpha};\alpha_0} = F_{\alpha;\alpha_0}\otimes\bar{F}_{\bar{\alpha};\alpha_0}$.

The inner products of highest weight vectors are subject to charge neutrality, i.e. they are non-vanishing only if the net charges in the two sectors are both zero. The normalization of the highest weight vectors must be in accordance with this condition and thus we have

$$\langle\alpha,\bar{\alpha};\alpha_0|\beta,\bar{\beta};\alpha_0\rangle = \kappa\delta_{\alpha,\beta}\delta_{\bar{\alpha},\bar{\beta}}. \tag{21}$$

In particular, the vacua are normalized as

$$\langle 0,0;\alpha_0|0,0;\alpha_0\rangle = \kappa. \tag{22}$$

In unitary theories the constant κ is usually positive and we normalize it to unity. As the Coulomb-gas system may well include non-unitary theories, κ can be negative. In that case we choose $\kappa = -1$. Thus we have

$${}_U\langle\alpha,\bar{\alpha};\alpha_0|\beta,\bar{\beta};\alpha_0\rangle_U = \delta_{\alpha,\beta}\delta_{\bar{\alpha},\bar{\beta}} \tag{23}$$

$${}_N\langle\alpha,\bar{\alpha};\alpha_0|\beta,\bar{\beta};\alpha_0\rangle_N = -\delta_{\alpha,\beta}\delta_{\bar{\alpha},\bar{\beta}}. \tag{24}$$

The subscripts U and N stand for unitary and non-unitary sectors, respectively. These two sectors have no intersection.

[3] Here we give the same background charge α_0 to both sectors. Even if we relax this condition and start by allocating different background charges α_0 and $\bar{\alpha}_0$ to holomorphic and antiholomorphic sectors, respectively, condition (26) restricts either $\alpha_0 = \pm\bar{\alpha}_0$. For $\alpha_0 = -\bar{\alpha}_0$ we have $\Omega = -1$ and $\alpha - \bar{\alpha} - 2\alpha_0 = 0$ instead of (29) and (30). This merely flips the sign of all antiholomorphic charges and thus does not give any new results.

2.2. Diffeomorphism invariant boundary states

Boundary states appearing in CFT are diffeomorphism invariant in the following sense [6, 24]. Let us consider a CFT on the upper half-plane $\mathrm{Im}\,\zeta \geqslant 0$, where ζ is a complex coordinate, $\zeta = x + \mathrm{i}y$, $x, y \in \mathbb{R}$. The boundary is $y = 0$, or $\zeta = \zeta^*$. Since the antiholomorphic coordinate dependence $\bar{\zeta}$ on the upper half-plane may be mapped onto the holomorphic dependence ζ^* on the lower half-plane [13], we often identify $\bar{\zeta}$ with ζ^*. Now, once the boundary is fixed, the conformal symmetry of the theory should be restricted so that the boundary is kept fixed. In other words, as the conformal transformation is generated by the energy–momentum tensor, the energy–momentum flow across the boundary must vanish,

$$[T(\zeta) - \bar{T}(\bar{\zeta})]_{\zeta=\bar{\zeta}} = 0. \tag{25}$$

This may be translated into a condition on boundary states by mapping a semiannular domain on the upper half-ζ-plane into a full annulus on the z-plane by $z = \exp(-2\pi\,\mathrm{i}\xi/L)$, $\xi = (T/\pi)\ln\zeta$. The boundary of the ζ-plane is mapped to the two concentric circles $|z| = 1, \exp(2\pi T/L)$ bordering the annulus on the z-plane. Since the z-plane allows radial quantization, (25) is written as the condition on the boundary states $|B\rangle$ (on $|z| = 1$),

$$(L_k - \bar{L}_{-k})|B\rangle = 0. \tag{26}$$

This condition, often called the Ishibashi condition, must then be satisfied by any boundary state in CFT.

We may follow the standard construction of boundary states in open string theory [25–27] and find boundary states on CBFS by starting from the coherent state ansatz

$${}_{\kappa}\langle B_{\alpha,\bar{\alpha};\alpha_0;\Omega}| = {}_{\kappa}\langle \alpha, \bar{\alpha}; \alpha_0| \prod_{k>0} \exp\left(-\frac{1}{k\Omega} a_k \bar{a}_k\right) \tag{27}$$

$$|B_{\alpha,\bar{\alpha};\alpha_0;\Omega}\rangle_{\kappa} = \prod_{k>0} \exp\left(-\frac{\Omega}{k} a_{-k} \bar{a}_{-k}\right) |\alpha, \bar{\alpha}; \alpha_0\rangle_{\kappa}. \tag{28}$$

The subscript κ is either U or N, specifying the normalization of the vacuum. As we have expressions of the Virasoro operators (11), (12) written in terms of the Heisenberg operators, one can see how the Virasoro modes L_n and $\bar{L}_{-n}$ operate on the coherent states $|B_{\alpha,\bar{\alpha};\alpha_0;\Omega}\rangle_{\kappa}$ by an explicit computation. It is shown [18] that condition (26) is satisfied if

$$\Omega = 1 \tag{29}$$

and

$$\alpha + \bar{\alpha} - 2\alpha_0 = 0. \tag{30}$$

Similarly, we see that ${}_{\kappa}\langle B_{\alpha,\bar{\alpha};\alpha_0;\Omega}|(L_n - \bar{L}_{-n}) = 0$ as long as (29) and (30) are satisfied. In the following we shall only consider such manifestly diffeomorphism invariant boundary states satisfying (29) and (30), and for simplicity we denote

$${}_{\kappa}\langle B(\alpha)| = {}_{\kappa}\langle B_{\alpha,2\alpha_0-\alpha;\alpha_0;\Omega=1}| \tag{31}$$

$$|B(\alpha)\rangle_{\kappa} = |B_{\alpha,2\alpha_0-\alpha;\alpha_0;\Omega=1}\rangle_{\kappa}. \tag{32}$$

We note that the sum of the (holomorphic + antiholomorphic) boundary charges agrees with the topological background charge on the Schottky double. Due to condition (30), an inner boundary (on the z-plane) contributes $2\alpha_0$ to the sum of the charges, and an outer boundary contributes $-2\alpha_0$. When we consider an annulus whose Schottky double is a torus, the sum of the boundary charges is zero ($2\alpha_0 - 2\alpha_0 = 0$), which coincides with the background charge of the torus expected from the Gauss–Bonnet theorem (the Euler number of a torus vanishes). For a disc, there is only an outer boundary which gives a charge $-2\alpha_0$. This agrees with the background charge of a sphere, which is the Schottky double of the disc. Thus, the geometry of the bulk can be assumed to be flat everywhere, since the curvature of the manifold is concentrated on the boundary.

2.3. Ishibashi states in free-field representation

As the basis of boundary states in CFT is normally spanned by Ishibashi states, we need to construct Ishibashi states in terms of our Fock space representation in order to translate the existing results of boundary CFT into the Coulomb-gas language. At least for the diagonal minimal models, it is shown [18] that the Ishibashi states are expressible as linear combinations of $|B(\alpha)\rangle_\kappa$, as far as partition functions on the cylinder are concerned.

Ishibashi states are defined for chiral representations of CFT and diagonalize the cylinder amplitudes (overlaps) to give characters,

$$\langle\!\langle i|(\tilde{q}^{1/2})^{L_0+\bar{L}_0-c/12}|j\rangle\!\rangle = \delta_{ij}\chi_j(\tilde{q}). \tag{33}$$

Here, we are considering a cylinder of length T and circumference L, or equivalently, an annulus on the z-plane with $1 \leqslant |z| \leqslant \exp(2\pi T/L)$. As the Hamiltonian is written as $H = (2\pi/L)(L_0 + \bar{L}_0 - c/12)$, the left-hand side of (33) is $\langle\!\langle i|\,\mathrm{e}^{-TH}|j\rangle\!\rangle$ with $\tilde{q} = \mathrm{e}^{-4\pi T/L}$. The characters of the minimal models are given by Rocha-Caridi [28],

$$\begin{aligned}\chi_{(r,s)}(q) &= \mathrm{Tr}_{(r,s)} q^{L_0-c/24}\\ &= \frac{\Theta_{pr-p's,pp'}(\tau) - \Theta_{pr+p's,pp'}(\tau)}{\eta(\tau)}\end{aligned} \tag{34}$$

where $\Theta_{\lambda,\mu}(\tau)$ and $\eta(\tau)$ are the Jacobi theta function and the Dedekind eta function, defined as $\Theta_{\lambda,\mu}(\tau) \equiv \sum_{k\in\mathbb{Z}} q^{(2\mu k+\lambda)^2/4\mu}$ and $\eta(\tau) \equiv q^{1/24}\prod_{n\geqslant 1}(1-q^n)$, with $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$. Thus, the cylinder amplitudes (33) are power series in $\tilde{q}$, divided by $\eta(\tilde{\tau})$.

As our boundary states $|B(\alpha)\rangle_\kappa$ are defined in a Fock space representation, we may explicitly compute the cylinder amplitudes between such states. They are [18]

$$\begin{aligned}{}_\kappa\langle B(\alpha)|\,\mathrm{e}^{-TH}|B(\beta)\rangle_\kappa &= {}_\kappa\langle B(\alpha)|(\tilde{q}^{1/2})^{L_0+\bar{L}_0-c/12}|B(\beta)\rangle_\kappa\\ &= \frac{\tilde{q}^{(\alpha-\alpha_0)^2}}{\eta(\tilde{\tau})}\kappa\,\delta_{\alpha,\beta}.\end{aligned} \tag{35}$$

The amplitudes between unitary and non-unitary sectors (e.g. ${}_U\langle B(\alpha)|\,\mathrm{e}^{-TH}|B(\beta)\rangle_N$) vanish because these sectors do not intersect. Note that $q^{(\alpha-\alpha_0)^2}/\eta(\tau)$ is the character $\chi_{\alpha;\alpha_0}(q)$ of the chiral CBFS $F_{\alpha;\alpha_0}$. In this sense, the state $|B(\alpha)\rangle_\kappa$ may be regarded as the Ishibashi state of $F_{\alpha;\alpha_0}$.

We may now compare expressions (33) and (35) to find a possible free-field representation of the Ishibashi states of minimal models. Defining

$$\langle\!\langle(r,s)| = {}_U\langle a_{r,s}| + {}_N\langle a_{r,-s}| \tag{36}$$

and

$$|(r,s)\rangle\!\rangle = |a_{r,s}\rangle_U + |a_{r,-s}\rangle_N \tag{37}$$

with

$${}_U\langle a_{r,s}| = \sum_{k\in\mathbb{Z}} {}_U\langle B(\alpha_{r,s} + k\sqrt{pp'})| \tag{38}$$

$${}_N\langle a_{r,-s}| = \sum_{k\in\mathbb{Z}} {}_N\langle B(\alpha_{r,-s} + k\sqrt{pp'})| \tag{39}$$

$$|a_{r,s}\rangle_U = \sum_{k\in\mathbb{Z}} |B(\alpha_{r,s} + k\sqrt{pp'})\rangle_U \tag{40}$$

$$|a_{r,-s}\rangle_N = \sum_{k\in\mathbb{Z}} |B(\alpha_{r,-s} + k\sqrt{pp'})\rangle_N \tag{41}$$

where $\alpha_{r,s}$ are given by (5), one can show that the states $|(r,s)\rangle\!\rangle$ diagonalize the overlaps and give minimal model characters

$$\langle\!\langle(r,s)|(\tilde{q}^{1/2})^{L_0+\bar{L}_0-c/12}|(r',s')\rangle\!\rangle = \delta_{rr'}\delta_{ss'}\chi_{(r,s)}(\tilde{q}). \tag{42}$$

The appearance of the non-unitary sector even in unitary CFTs may seem odd, but this is necessary to describe cylinder diagrams where otherwise unphysical states would propagate. On the disc the non-unitary sector decouples and does not contribute to correlation functions (section 3). The states (36), (37) are a good candidate for the minimal model Ishibashi states as far as the modular properties are concerned. We, however, immediately note that such 'Ishibashi' states are not unique since $|(r,s)\rangle$ and $|(p'-r, p-s)\rangle$ give rise to the same character but are perpendicular to each other. In order to have a unique Ishibashi state for each primary field $(r,s) \sim (p'-r, p-s)$ of minimal models, we define the symmetrized states

$$\begin{aligned}\langle\!\langle\phi_{r,s}| &= \langle\!\langle\phi_{p'-r,p-s}| \\ &= \tfrac{1}{\sqrt{2}}(\langle\!\langle(r,s)| + \langle\!\langle(p'-r, p-s)|)\end{aligned} \tag{43}$$

$$\begin{aligned}|\phi_{r,s}\rangle\!\rangle &= |\phi_{p'-r,p-s}\rangle\!\rangle \\ &= \tfrac{1}{\sqrt{2}}(|(r,s)\rangle\!\rangle + |(p'-r, p-s)\rangle\!\rangle)\end{aligned} \tag{44}$$

and shall regard them as our free-field realization of the Ishibashi states. Although this 'symmetrization' was not considered in [18], such a prescription to ensure the equivalence of $(r,s) \sim (p'-r, p-s)$ is necessary.

2.4. Cardy's consistent boundary states

Physical boundary states in CFT are not only diffeomorphism invariant, but must also satisfy an extra constraint called Cardy's consistency condition. We consider a cylinder of length T and circumference L as before and assume that boundary conditions $\tilde{\alpha}$ and $\tilde{\beta}$ are imposed on the two boundaries. Then, depending on how we define the direction of time, the partition function on this cylinder may be written in two different ways. We may first regard the cylinder as an open string propagating in the periodic direction of time, with boundary conditions $\tilde{\alpha}$ and $\tilde{\beta}$ imposed at the two ends of the string. The partition function is then a sum of the chiral characters, $Z_{\tilde{\alpha}\tilde{\beta}}(q) = \sum_j n^j_{\tilde{\alpha}\tilde{\beta}}\chi_j(q)$, where $\chi_j(q)$ is the character for the chiral representation j and $n^j_{\tilde{\alpha}\tilde{\beta}}$ is a non-negative integer representing the multiplicity of the representations. We have defined $q = \mathrm{e}^{-\pi L/T}$. We may also see the cylinder as a closed string propagating from one boundary (with boundary condition $\tilde{\beta}$) to the other (with $\tilde{\alpha}$). Then the partition function is simply the cylinder amplitude between the two boundaries, $\langle\tilde{\alpha}|\,\mathrm{e}^{-TH}|\tilde{\beta}\rangle$. Due to the equivalence of the two pictures, we have $\sum_j n^j_{\tilde{\alpha}\tilde{\beta}}\chi_j(q) = \langle\tilde{\alpha}|(\tilde{q}^{1/2})^{L_0+\bar{L}_0-c/12}|\tilde{\beta}\rangle$. This is the consistency condition which needs to be satisfied by the boundary states $\langle\tilde{\alpha}|$ and $|\tilde{\beta}\rangle$.

If we have an appropriate basis of the boundary states, the right-hand side of the consistency equation may be expanded using the basis states $\{\langle a|\}, \{|b\rangle\}$ as

$$\sum_j n^j_{\tilde{\alpha}\tilde{\beta}}\chi_j(q) = \sum_{a,b}\langle\tilde{\alpha}|a\rangle\langle a|(\tilde{q}^{1/2})^{L_0+\bar{L}_0-c/12}|b\rangle\langle b|\tilde{\beta}\rangle. \tag{45}$$

Solving this equation, the *consistent* boundary states $\{|\tilde{\alpha}\rangle\}$ are found as linear combinations of the basis states. It is convenient to use the Ishibashi states for such a basis. For diagonal minimal models, using the property (33) of the Ishibashi states and the modular transformation of the characters $\chi_i(q) = \sum_j S_{ij}\chi_j(\tilde{q})$ under $\tau \to \tilde{\tau} = -1/\tau$, we have, by equating the coefficients of the characters,

$$\sum_i n^i_{\tilde{\alpha}\tilde{\beta}} S_{ij} = \langle\tilde{\alpha}|j\rangle\!\rangle\langle\!\langle j|\tilde{\beta}\rangle. \tag{46}$$

Assuming the existence of a state $|\tilde{0}\rangle$ such that $n^i_{\tilde{0}\tilde{\alpha}} = n^i_{\tilde{\alpha}\tilde{0}} = \delta^i_{\tilde{\alpha}}$, (46) was solved by Cardy [6] as

$$|\tilde{\alpha}\rangle = \sum_j |j\rangle\!\rangle\langle\!\langle j|\tilde{\alpha}\rangle = \sum_j \frac{S_{\alpha j}}{\sqrt{S_{0j}}}|j\rangle\!\rangle. \tag{47}$$

Now, as the minimal model Ishibashi states have been written in the free-field representation (43), (44), Cardy's consistent boundary states (47) can also be expressed using our coherent boundary states by substituting (44) into (47).

For the convenience of later discussions, let us spell out these in the specific example of the Ising model. The Ising model is the simplest non-trivial minimal model having the central charge $c = 1/2$ and is characterized by the two coprime integers $p = 4$ and $p' = 3$. There are three operators, the identity I, the energy ϵ and the spin σ, having the conformal dimensions 0, 1/2 and 1/16, respectively, and are identified in the Kac table as $I = \phi_{1,1} = \phi_{2,3}$, $\epsilon = \phi_{2,1} = \phi_{1,3}$ and $\sigma = \phi_{1,2} = \phi_{2,2}$. Using the modular transformation matrices for the characters, Cardy's boundary states are written as [6]

$$|\tilde{I}\rangle = |\tilde{0}\rangle = 2^{-1/2}|I\rangle\!\rangle + 2^{-1/2}|\epsilon\rangle\!\rangle + 2^{-1/4}|\sigma\rangle\!\rangle \tag{48}$$

$$|\tilde{\epsilon}\rangle = 2^{-1/2}|I\rangle\!\rangle + 2^{-1/2}|\epsilon\rangle\!\rangle - 2^{-1/4}|\sigma\rangle\!\rangle \tag{49}$$

$$|\tilde{\sigma}\rangle = |I\rangle\!\rangle - |\epsilon\rangle\!\rangle. \tag{50}$$

It is argued that the first two states correspond to the fixed (up and down) boundary conditions since they differ only in the sign of $|\sigma\rangle\!\rangle$ which is associated with the spin operator. The last state (50) is then identified as the free boundary state. Using (37) and (44), these states are written in our free-field representation as

$$\begin{aligned}|\tilde{I}\rangle = 2^{-1}(&|a_{1,1}\rangle_U + |a_{1,-1}\rangle_N + |a_{2,3}\rangle_U + |a_{2,-3}\rangle_N + |a_{2,1}\rangle_U + |a_{2,-1}\rangle_N + |a_{1,3}\rangle_U + |a_{1,-3}\rangle_N)\\ &+ 2^{-3/4}(|a_{1,2}\rangle_U + |a_{1,-2}\rangle_N + |a_{2,2}\rangle_U + |a_{2,-2}\rangle_N)\end{aligned} \tag{51}$$

$$\begin{aligned}|\tilde{\epsilon}\rangle = 2^{-1}(&|a_{1,1}\rangle_U + |a_{1,-1}\rangle_N + |a_{2,3}\rangle_U + |a_{2,-3}\rangle_N + |a_{2,1}\rangle_U + |a_{2,-1}\rangle_N + |a_{1,3}\rangle_U + |a_{1,-3}\rangle_N)\\ &- 2^{-3/4}(|a_{1,2}\rangle_U + |a_{1,-2}\rangle_N + |a_{2,2}\rangle_U + |a_{2,-2}\rangle_N)\end{aligned} \tag{52}$$

$$\begin{aligned}|\tilde{\sigma}\rangle = 2^{-1/2}(&|a_{1,1}\rangle_U + |a_{1,-1}\rangle_N + |a_{2,3}\rangle_U + |a_{2,-3}\rangle_N - |a_{2,1}\rangle_U\\ &- |a_{2,-1}\rangle_N - |a_{1,3}\rangle_U - |a_{1,-3}\rangle_N)\end{aligned} \tag{53}$$

where the states on the right-hand sides are defined by (40), (41). They are superpositions of countably many coherent states with different boundary charges.

3. Boundary correlation functions

Now let us discuss how to compute boundary correlation functions in our Coulomb-gas picture. After giving the general formalism, we shall focus on the one-point function of $\phi_{r,s}(z,\bar{z})$ and

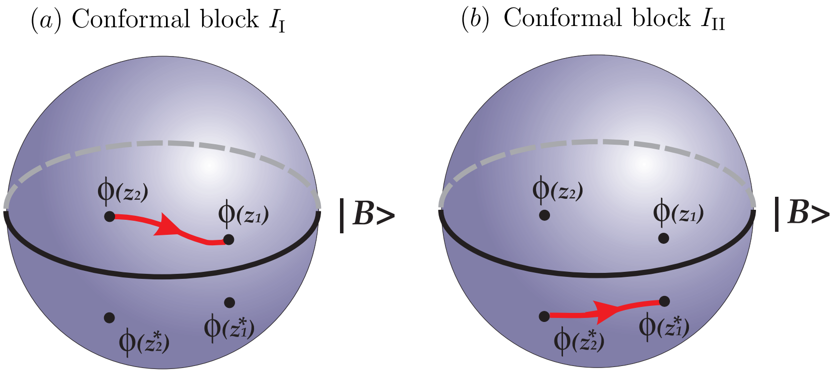

Figure 1. The two conformal blocks $I_{\rm I}$ and $I_{\rm II}$ in the boundary two-point function of $\phi_{1,2}$ on the disc. The sphere represents the double of the disc, and the upper and lower hemispheres stand for the holomorphic and antiholomorphic sectors, which are glued on the boundary (the equator). The lower hemisphere coordinates z_i^* are obtained from $\bar{z}_i$ by the doubling $(z_i, \bar{z}_i) \to (z_i, z_i^*)$. In the case of $I_{\rm I}$ where the screening operator Q_- lies in the holomorphic sector, the integration contour can be deformed into the Pochhammer type around z_1 and z_2. The integral is then proportional to the one from z_2 to z_1 (*a*). Similarly, the screening contour of $I_{\rm II}$ is in the antiholomorphic sector and the integral is proportional to the one from z_2^* to z_1^* (*b*).

the two-point function of $\phi_{1,2}(z, \bar{z})$, and derive their explicit expressions on the unit disc. Once correlators on the disc are obtained, it is straightforward to map them on the half-plane. At the end of this section, we compare our free-field approach and OPE computation of boundary correlation functions.

3.1. Screened vertex operators and BRST states

In the CBFS language, correlation functions on the full plane are described as follows [14]. We define chiral screened vertex operators $V_{r,s}^{m,n}(z)$ as

$$V_{r,s}^{m,n}(z) = \oint \prod_{i=1}^{m} \mathrm{d}u_i \prod_{j=1}^{n} \mathrm{d}v_j \, V_{r,s}(z) V_+(u_1) \cdots V_+(u_m) V_-(v_1) \cdots V_-(v_n) \quad (54)$$

where for conciseness we have denoted $V_{\alpha_{r,s}}(z)$ as $V_{r,s}(z)$ and $V_{\alpha_\pm}(z)$ as $V_\pm(z)$, and the integration contours are those of Felder's, all going through z and encircling the origin (figure 1 of [14]). Such an operator is a primary field of conformal dimension $h_{r,s}$. We also denote the chiral CBFS $F_{\alpha_{r,s};\alpha_0}$ as $F_{r,s}$. The operator $V_{r,s}^{m,n}(z)$ defines a map from one Fock space to another,

$$V_{r,s}^{m,n}(z) : F_{r_0,s_0} \to F_{r_0+r-2m-1,s_0+s-2n-1}. \quad (55)$$

The p-point correlator

$$\langle 0; \alpha_0 | V_{r_1,s_1}^{m_1,n_1}(z_1) \cdots V_{r_p,s_p}^{m_p,n_p}(z_p) | 0; \alpha_0 \rangle \quad (56)$$

is then seen as a sequence of mappings,

$$F_{1,1} \to F_{r_p-2m_p,s_p-2n_p} \to \cdots \quad (57)$$

and the final state $V_{r_1,s_1}^{m_1,n_1}(z_1) \cdots V_{r_p,s_p}^{m_p,n_p}(z_p)|0; \alpha_0\rangle$ must belong to $F_{1,1}$ in order to have a non-trivial inner product with $\langle 0; \alpha_0| \in F_{1,1}^*$ (the dual module of $F_{1,1}$). The same p-point correlator may be expressed differently as

$$\langle \alpha_{p'-1,p-1}; \alpha_0 | V_{r_1,s_1}^{m_1',n_1'}(z_1) \cdots V_{r_p,s_p}^{m_p',n_p'}(z_p) | 0; \alpha_0 \rangle \quad (58)$$

but this is in fact proportional to (56). A key object in this formalism is the BRST operator

$$Q_r = \frac{e^{2\pi i\alpha_+^2 r} - 1}{r\left(e^{2\pi i\alpha_+^2} - 1\right)} \oint \prod_{i=1}^{r} du_i\, V_+(u_1)\cdots V_+(u_r) \tag{59}$$

which maps $F_{r,s}$ to $F_{-r,s}$. The BRST operator is nilpotent, $Q_r Q_{p'-r} = 0$, and physical states are realized as the cohomology space, Ker $Q_r/\mathrm{Im}\, Q_{p'-r}$. An important point which is evident in this picture is that all intermediate states appearing in the correlator are BRST states since the vacuum $|0;\alpha_0\rangle$ is a BRST state and the screened vertex operators map BRST states to BRST states [14].

We shall combine the above machinery and the boundary states of the previous section to compute correlation functions on the unit disc. The in-state of the correlators is the non-chiral vacuum (in the unitary sector) at the origin, $|0\rangle = |0, 0; \alpha_0\rangle_U$, which is a BRST state. For the out-state, we choose a boundary state ${}_U\langle B(\alpha)|$ of (31) with a fixed boundary charge α. We only consider the unitary sector since the non-unitary sector does not give non-trivial inner products with the in-state vacuum. The correlators are then obtained by inserting non-chiral screened vertex operators

$$\mathcal{V}^{(m_i,n_i),(\bar{m}_i,\bar{n}_i)}_{(r_i,s_i),(\bar{r}_i,\bar{s}_i)}(z_i,\bar{z}_i) = V^{m_i,n_i}_{r_i,s_i}(z_i)\bar{V}^{\bar{m}_i,\bar{n}_i}_{\bar{r}_i,\bar{s}_i}(\bar{z}_i) \tag{60}$$

between ${}_U\langle B(\alpha)|$ and $|0, 0; \alpha_0\rangle_U$. As we focus on diagonal theories, the antiholomorphic vertex operators have the same conformal dimensions as the holomorphic counterparts, i.e., either $(\bar{r}_i, \bar{s}_i) = (r_i, s_i)$ or $(\bar{r}_i, \bar{s}_i) = (p' - r_i, p - s_i)$. Note that, in this construction, all the intermediate states are manifestly BRST invariant because no spurious states arise. Actual boundary p-point correlation functions for physical boundary conditions are obtained by summing the fixed boundary-charge correlators,

$${}_U\langle B(\alpha)| \prod_{i=1}^{p} V^{m_i,n_i}_{r_i,s_i}(z_i)\bar{V}^{\bar{m}_i,\bar{n}_i}_{\bar{r}_i,\bar{s}_i}(\bar{z}_i)|0, 0; \alpha_0\rangle_U \tag{61}$$

over the boundary charges according to the linear combinations (such as (51)–(53)) given by Cardy's method. Note that (61) is non-vanishing only for certain configurations of screening charges. The net holomorphic and antiholomorphic charges are respectively,

$$-\alpha + \sum_i \alpha_{r_i,s_i} + \sum_i m_i\alpha_+ + \sum_i n_i\alpha_- \tag{62}$$

and

$$\alpha - 2\alpha_0 + \sum_i \alpha_{\bar{r}_i,\bar{s}_i} + \sum_i \bar{m}_i\alpha_+ + \sum_i \bar{n}_i\alpha_- \tag{63}$$

which must vanish independently. These charge neutrality conditions associate the allowed values of α with the numbers of holomorphic and antiholomorphic screening operators.

In the computation of expression (61), $V_{r,s}(z)\bar{V}_{r,s}(\bar{z})$, $V_{r,s}(z)\bar{V}_{p'-r,p-s}(\bar{z})$, $V_{p'-r,p-s}(z)\bar{V}_{r,s}(\bar{z})$ and $V_{p'-r,p-s}(z)\bar{V}_{p'-r,p-s}(\bar{z})$ all correspond to a non-chiral field $\phi_{r,s}(z,\bar{z})$ and one may use any combinations. This is ensured by the fact that (61) is essentially a chiral $2p$-point function where the equivalence of $(r, s) \leftrightarrow (p' - r, p - s)$ (after the truncation of unphysical states) is guaranteed [14, 21]. In particular, we are allowed to use $V_{r,s}(z)$ as the holomorphic and $\bar{V}_{p'-r,p-s}(\bar{z})$ as the antiholomorphic ('mirror image') part of a single non-chiral operator (the analogy of a mirror and a boundary of CFT is based on the conformal Ward identity [13] which does not distinguish $\bar{V}_{r,s}(\bar{z})$ from $\bar{V}_{p'-r,p-s}(\bar{z})$). Due to this equivalence, apparently different choices of vertex operators should all lead to a same conformal block function. In practice, similar to the case of the Coulomb-gas computation without boundary, we shall choose such vertex operators that the number of screening operators is minimized.

In the following subsections we shall evaluate expression (61) for particular cases of one- and two-point functions.

3.2. Boundary one-point functions

For evaluation of the boundary one-point correlator

$$ {}_U\langle B(\alpha)|V_{r,s}^{m,n}(z)\bar{V}_{\bar{r},\bar{s}}^{\bar{m},\bar{n}}(\bar{z})|0,0;\alpha_0\rangle_U \tag{64}$$

it is convenient to choose $(\bar{r},\bar{s}) = (p'-r, p-s)$ (the other choice $(\bar{r},\bar{s}) = (r,s)$ should lead to the same result but involves complicated integral expressions). According to (55), the holomorphic part of the CBFS is mapped as

$$ F_{1,1} \to F_{r-2m,s-2n} \tag{65}$$

and the antiholomorphic part is mapped as

$$ \bar{F}_{1,1} \to \bar{F}_{p'-r-2\bar{m},p-s-2\bar{n}}. \tag{66}$$

Since ${}_U\langle B(\alpha)| \in F^*_{\alpha;\alpha_0} \otimes F^*_{2\alpha_0-\alpha;\alpha_0}$, the correlator is non-vanishing only when $F_{\alpha;\alpha_0} = F_{r-2m,s-2n}$ and $\bar{F}_{2\alpha_0-\alpha;\alpha_0} = \bar{F}_{p'-r-2\bar{m},p-s-2\bar{n}}$, that is,

$$ \alpha = \tfrac{1}{2}(1-r+2m)\alpha_+ + \tfrac{1}{2}(1-s+2n)\alpha_- \tag{67}$$

and

$$ 2\alpha_0 - \alpha = \tfrac{1}{2}(1-p'+r+2\bar{m})\alpha_+ + \tfrac{1}{2}(1-p+s+2\bar{n})\alpha_-. \tag{68}$$

Summing (67), (68) and using $\alpha_+ = \sqrt{p/p'}, \alpha_- = -\sqrt{p'/p}$, we have $(m+\bar{m})\alpha_+ + (n+\bar{n})\alpha_- = 0$, or

$$ m+\bar{m} = 0 \qquad n+\bar{n} = 0 \tag{69}$$

implying no screening charges. Then from (67) we have $\alpha = \alpha_{r,s}$, and the correlator (64) is evaluated as

$$ {}_U\langle B(\alpha_{r,s})|V_{r,s}^{0,0}(z)\bar{V}_{\bar{r},\bar{s}}^{0,0}(\bar{z})|0,0;\alpha_0\rangle_U = (1-z\bar{z})^{-2h} \tag{70}$$

where $h = \alpha_{r,s}(\alpha_{r,s}-2\alpha_0)$. As we shall see later, this is proportional to the two-point correlator on the full plane.

3.3. Boundary two-point functions of $\phi_{1,2}$

As a less trivial case, we consider the two-point correlator of the primary field $\phi_{1,2}$. For the convenience of calculation we define one of the operators as $\phi_{1,2}(z_1,\bar{z}_1) = V_{1,2}(z_1)\bar{V}_{p'-1,p-2}(\bar{z}_1)$ and the other as $\phi_{1,2}(z_2,\bar{z}_2) = V_{1,2}(z_2)\bar{V}_{1,2}(\bar{z}_2)$. Expression (61) then becomes

$$ {}_U\langle B(\alpha)|V_{1,2}^{m_1,n_1}(z_1)\bar{V}_{p'-1,p-2}^{\bar{m}_1,\bar{n}_1}(\bar{z}_1)V_{1,2}^{m_2,n_2}(z_2)\bar{V}_{1,2}^{\bar{m}_2,\bar{n}_2}(\bar{z}_2)|0,0;\alpha_0\rangle_U. \tag{71}$$

In the holomorphic and antiholomorphic sectors, the CBFSs are mapped as

$$ F_{1,1} \to F_{1-2m_2,2-2n_2} \to F_{1-2m_1-2m_2,3-2n_1-2n_2} \tag{72}$$

$$ \bar{F}_{1,1} \to \bar{F}_{1-2\bar{m}_2,2-2\bar{n}_2} \to \bar{F}_{p'-2\bar{m}_1-2\bar{m}_2-1,p-2\bar{n}_1-2\bar{n}_2-1} \tag{73}$$

and hence, in order that the correlator be non-vanishing we must have

$$ \alpha = (m_1+m_2)\alpha_+ + (n_1+n_2-1)\alpha_- \tag{74}$$

$$ 2\alpha_0 - \alpha = (\bar{m}_1+\bar{m}_2+1)\alpha_+ + (\bar{n}_1+\bar{n}_2+1)\alpha_-. \tag{75}$$

Adding the above two expressions we have $(m + \bar{m})\alpha_+ + (n + \bar{n} - 1)\alpha_- = 0$, where $m = m_1 + m_2$ and $n = n_1 + n_2$ are the numbers of positive (α_+) and negative (α_-) screening operators in the holomorphic sector, and similarly $\bar{m} = \bar{m}_1 + \bar{m}_2$ and $\bar{n} = \bar{n}_1 + \bar{n}_2$ for the antiholomorphic counterparts. This charge neutrality condition implies

$$m + \bar{m} = 0 \qquad n + \bar{n} = 1. \tag{76}$$

Then we have two possibilities:

$$\text{(I)} \quad m = \bar{m} = \bar{n} = 0 \qquad n = 1 \tag{77}$$

$$\text{(II)} \quad m = \bar{m} = n = 0 \qquad \bar{n} = 1. \tag{78}$$

From (74), we have the boundary charge $\alpha = \alpha_{1,1} = 0$ for (I) and $\alpha = \alpha_{1,3} = -\alpha_-$ for (II).

For the charge configuration (I), we have one screening operator

$$Q_- = \oint \mathrm{d}v V_-(v) \tag{79}$$

in the holomorphic sector and thus the correlator (71) takes the form

$$I_{\mathrm{I}} = {}_U\langle B(\alpha_{1,1})| \oint \mathrm{d}v V_{1,2}(z_1)\bar{V}_{p'-1,p-2}(\bar{z}_1)V_-(v)V_{1,2}(z_2)\bar{V}_{1,2}(\bar{z}_2)|0, 0; \alpha_0\rangle_U \tag{80}$$

which, by an explicit calculation, reduces to

$$\oint \mathrm{d}v(1 - z_1\bar{z}_1)^a(1 - z_1\bar{z}_2)^b(1 - v\bar{z}_1)^c(1 - v\bar{z}_2)^d(1 - z_2\bar{z}_1)^a(1 - z_2\bar{z}_2)^b(z_1 - v)^d$$
$$\times (z_1 - z_2)^b(v - z_2)^d(\bar{z}_1 - \bar{z}_2)^a \tag{81}$$

with $a = 2\alpha_{1,2}(2\alpha_0 - \alpha_{1,2})$, $b = 2\alpha_{1,2}^2$, $c = 2\alpha_-(2\alpha_0 - \alpha_{1,2})$ and $d = 2\alpha_-\alpha_{1,2}$. As this expression is analytic, we may deform the integration contour as long as it is closed and non-contractible. In this case the screening operator must lie entirely on the holomorphic part and Felder's contour can be deformed into the Pochhammer type, going around z_1 and z_2. The correlator is then proportional to the integration from z_2 to z_1, and is written as

$$I_{\mathrm{I}} = \mathcal{N}_{\mathrm{I}}(1 - z_1\bar{z}_1)^a(1 - z_1\bar{z}_2)^b(1 - z_2\bar{z}_1)^a(1 - z_2\bar{z}_2)^b(z_1 - z_2)^b(\bar{z}_1 - \bar{z}_2)^a$$
$$\times \int_{z_2}^{z_1} \mathrm{d}v(1 - v\bar{z}_1)^c(1 - v\bar{z}_2)^d(z_1 - v)^d(v - z_2)^d. \tag{82}$$

Here, $\mathcal{N}_{\mathrm{I}}$ is a constant arising from the deformation of the contour. Note that, at this point, the expression is similar (in fact, proportional) to the integral representation of a chiral four-point conformal block [21] (without boundary). One may then proceed in the standard manner, namely, by fixing the projective $SL(2, \mathbb{C})$ gauge, performing the integration and then recovering the coordinate dependence. We thus have

$$I_{\mathrm{I}} = \mathcal{N}_{\mathrm{I}}(1 - z_1\bar{z}_1)^a(1 - z_2\bar{z}_2)^a[\eta(\eta - 1)]^a \frac{\Gamma(1 - \alpha_-^2)^2}{\Gamma(2 - 2\alpha_-^2)} F(2a, 1 - \alpha_-^2, 2 - 2\alpha_-^2; \eta) \tag{83}$$

where $F = {}_2F_1$ is the hypergeometric function of the Gaussian type, and η is defined as

$$\eta = \frac{(z_1 - z_2)(\bar{z}_2 - \bar{z}_1)}{(1 - z_1\bar{z}_1)(1 - z_2\bar{z}_2)}. \tag{84}$$

The calculation for (II) goes similarly. As we have one screening operator

$$\bar{Q}_- = \oint \mathrm{d}\bar{v}\bar{V}_-(\bar{v}) \tag{85}$$

in the antiholomorphic sector, the correlator is written as

$$I_{\rm II} = {}_U\langle B(\alpha_{1,3})| \oint {\rm d}\bar{v} V_{1,2}(z_1)\bar{V}_{p'-1,p-2}(\bar{z}_1)\bar{V}_-(\bar{v})V_{1,2}(z_2)\bar{V}_{1,2}(\bar{z}_2)|0,0;\alpha_0\rangle_U$$
$$= \mathcal{N}_{\rm II}(z_1-z_2)^b(\bar{z}_1-\bar{z}_2)^a(1-z_1\bar{z}_1)^a(1-z_1\bar{z}_2)^b(1-z_2\bar{z}_1)^a(1-z_2\bar{z}_2)^b$$
$$\times \int_{\bar{z}_2}^{\bar{z}_1} {\rm d}\bar{v}(\bar{z}_1-\bar{v})^c(\bar{v}-\bar{z}_2)^d(1-z_1\bar{v})^d(1-z_2\bar{v})^d. \qquad (86)$$

We have again deformed Felder's integration contour (this time in the antiholomorphic sector) into the Pochhammer contour around $\bar{z}_1$ and $\bar{z}_2$. The resulting integral is proportional to the one from $\bar{z}_2$ to $\bar{z}_1$, and $\mathcal{N}_{\rm II}$ is a constant. Performing the integration we have

$$I_{\rm II} = \mathcal{N}_{\rm II}(1-z_1\bar{z}_1)^a(1-z_2\bar{z}_2)^a[\eta(\eta-1)]^a(-\eta)^{b-a}\frac{\Gamma(1-\alpha_-^2)\Gamma(3\alpha_-^2-1)}{\Gamma(2\alpha_-^2)}$$
$$\times F(\alpha_-^2, 1-\alpha_-^2, 2\alpha_-^2;\eta). \qquad (87)$$

The two correlators $I_{\rm I}$ and $I_{\rm II}$ with fixed boundary charges are represented schematically (in the Schottky double picture) in figure 1. They correspond to the two conformal blocks of the chiral four-point function.

3.4. *Correlation functions on the half-plane*

Boundary correlation functions obtained on the unit disc are mapped on the half-plane by the global conformal transformation

$$w = -{\rm i}y_0\frac{z-1}{z+1} \qquad \bar{w} = {\rm i}y_0\frac{\bar{z}-1}{\bar{z}+1} \qquad (88)$$

which takes the unit circle $|z| = 1$ on the z-plane to the infinite line ${\rm Im}\, w = 0$ on the w-plane, and the origin $z = 0$ to the point $w = {\rm i}y_0$, $y_0 \in \mathbb{R}$. Under this transformation the holomorphic coordinate dependence on the unit disc is mapped onto the upper half-w-plane, and the antiholomorphic dependence is mapped onto the lower half-$\bar{w}$-plane. Using the transformation (88), p-point correlation functions on the half-plane are written using those on the z-plane, i.e. on the disc, as

$$\langle\phi_1(w_1,\bar{w}_1)\cdots\phi_p(w_p,\bar{w}_p)\rangle_{\rm UHP} = \prod_{i=1}^p\left(\frac{{\rm d}w_i}{{\rm d}z_i}\right)^{-h_i}\left(\frac{{\rm d}\bar{w}_i}{{\rm d}\bar{z}_i}\right)^{-\bar{h}_i}\langle\phi_1(z_1,\bar{z}_1)\cdots\phi_p(z_p,\bar{z}_p)\rangle_{\rm disc}$$
$$= \prod_{i=1}^p\left\{\frac{2y_0}{(z_i+1)(\bar{z}_i+1)}\right\}^{-2h_i}\langle\phi_1(z_1,\bar{z}_1)\cdots\phi_p(z_p,\bar{z}_p)\rangle_{\rm disc} \qquad (89)$$

where we have assumed $h_i = \bar{h}_i$. The parameter η of (84) is mapped under this transformation as

$$\eta = \frac{(z_1-z_2)(\bar{z}_2-\bar{z}_1)}{(1-z_1\bar{z}_1)(1-z_2\bar{z}_2)} = \frac{(w_1-w_2)(\bar{w}_1-\bar{w}_2)}{(w_1-\bar{w}_1)(w_2-\bar{w}_2)} \qquad (90)$$

which is an anharmonic ratio of the four points w_1, w_2, $\bar{w}_1$ and $\bar{w}_2$.

Now, the boundary one-point function of $\phi_{r,s}$ on the upper half-plane is easily found by using (70) as

$$\langle\phi_{r,s}(w,\bar{w})\rangle_{B(\alpha_{r,s})} = \left\{\frac{2y_0(1-z\bar{z})}{(z+1)(\bar{z}+1)}\right\}^{-2h}$$
$$= [-{\rm i}(w-\bar{w})]^{-2h}$$
$$= (2y)^{-2h} \qquad (91)$$

where $h = h_{r,s} = \alpha_{r,s}(\alpha_{r,s} - 2\alpha_0)$ is the conformal dimension of the operator $\phi_{r,s}$, and $w = x + \mathrm{i}y$, $\bar{w} = w^* = x - \mathrm{i}y$, $x, y \in \mathbb{R}$. Note that the result is y_0 independent. The two-point function of $\phi_{1,2}$ on the disc is mapped onto the half-plane similarly. For the conformal block (I) we have

$$\langle\phi_{1,2}(w_1,\bar{w}_1)\phi_{1,2}(w_2,\bar{w}_2)\rangle_{B(\alpha_{1,1})} = \left\{\frac{4y_0^2}{(z_1+1)(\bar{z}_1+1)(z_2+1)(\bar{z}_2+1)}\right\}^{-2h} I_{\mathrm{I}}$$
$$= \mathcal{N}_{\mathrm{I}}\left\{\frac{(w_1-\bar{w}_1)(\bar{w}_2-w_2)}{(w_1-w_2)(\bar{w}_1-\bar{w}_2)(w_1-\bar{w}_2)(\bar{w}_1-w_2)}\right\}^{2h}$$
$$\times\frac{\Gamma(1-\alpha_-^2)^2}{\Gamma(2-2\alpha_-^2)}F(-4h, 1-\alpha_-^2, 2-2\alpha_-^2;\eta) \tag{92}$$

and for (II) we have

$$\langle\phi_{1,2}(w_1,\bar{w}_1)\phi_{1,2}(w_2,\bar{w}_2)\rangle_{B(\alpha_{1,3})} = \left\{\frac{4y_0^2}{(z_1+1)(\bar{z}_1+1)(z_2+1)(\bar{z}_2+1)}\right\}^{-2h} I_{\mathrm{II}}$$
$$= \mathcal{N}_{\mathrm{II}}\left\{\frac{(w_1-\bar{w}_1)(\bar{w}_2-w_2)}{(w_1-w_2)(\bar{w}_1-\bar{w}_2)(w_1-\bar{w}_2)(\bar{w}_1-w_2)}\right\}^{2h}$$
$$\times\frac{\Gamma(1-\alpha_-^2)\Gamma(3\alpha_-^2-1)}{\Gamma(2\alpha_-^2)}(-\eta)^{2h+2\alpha_-^2}F(\alpha_-^2, 1-\alpha_-^2, 2\alpha_-^2;\eta) \tag{93}$$

where $h = h_{1,2} = \alpha_{1,2}(\alpha_{1,2} - 2\alpha_0)$. Physical correlation functions are linear sums of these conformal blocks where the coefficients are given by Cardy's states.

3.5. Boundary states and conformal blocks

Before illustrating in specific examples, we mention how the above description of boundary correlation functions fits into the conventional discussion of [7, 8], and see the validity and limitation of the Coulomb-gas approach.

The charge neutrality conditions (67), (68) for a one-point function pick up the coefficient of the corresponding Ishibashi state from a Cardy state, since $\alpha = \alpha_{r,s}$ is the only boundary charge which gives a non-vanishing term. This agrees with our understanding that the coefficients of Cardy's state are essentially the one-point coupling constants of bulk (closed string vertex) operators to the boundary (brane) [7, 8]. Once one-point coupling constants are known, it is in principle possible to compute boundary multipoint functions since they reduce to one-point functions after repeated use of bulk OPEs (figure 2(*a*)). In particular, we may start such a procedure from the farthest point from the boundary (in the radial ordering sense), approaching the boundary by performing OPE with the farthest remaining point one by one. Due to naturality of CFT, such OPEs are translated into the fusions of operators,

$$\begin{aligned}
[\phi_1]\times[\phi_2] &= [j_1]\\
[j_1]\times[\phi_3] &= [j_2]\\
&\cdots\\
[j_{p-2}]\times[\phi_p] &= [j_{p-1}]
\end{aligned} \tag{94}$$

which define a conformal block with no subchains (figure 2(*b*)). As the fusion of primary operators is equivalent to the map (55) between CBFSs restricted to BRST subspaces [14], the conformal block of figure 2(*b*) is represented by our fixed boundary-charge correlator (61) with α corresponding to $[j_{p-1}]$. The Ishibashi state $\langle\langle j_{p-1}|$ acts as a filter (or a half-mirror) transmitting only the Virasoro representation $[j_{p-1}]$. This property of Ishibashi states is captured by boundary charges and charge neutrality.

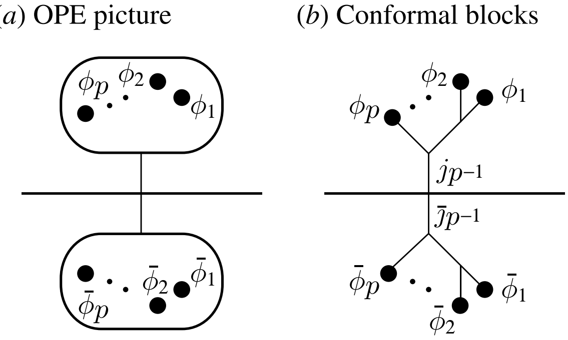

Figure 2. OPE and conformal blocks of boundary p-point functions. Repeating OPEs in the bulk, the boundary correlation function reduces to one-point functions (a). This can also be seen as conformal blocks with internal channels $j_{i<p-1}$ and boundary channel j_{p-1} (b).

Due to the absence of internal channels, two-point functions on the disc (or half-plane) are completely determined by a boundary state, apart from the normalization of conformal blocks. In the case of p-point functions with $p \geqslant 3$, however, states in the internal channels $j_{i<p-1}$ cannot in general be determined uniquely even if the state in the 'boundary channel' j_{p-1} is fixed (an example is the spin three-point function of the Ising model with $j_2 = \sigma$, where j_1 can be I or ϵ). Corresponding to this, the contours of (61) with $p \geqslant 3$ may be deformed in several different ways to give independent convergent functions which are expected to reproduce conformal blocks with different internal states. The relative coefficients of such conformal blocks in a boundary correlation function cannot be determined by the boundary state (as these conformal blocks belong to the same Ishibashi state) but should be constrained by information of the bulk.

4. Ising model

We shall illustrate the method presented in the previous section in the example of the critical Ising model. Before starting the actual calculations we note from the discussions of the previous section that the non-unitary sector and the boundary states with charges outside the Kac table do not contribute to correlation functions. Neglecting such unnecessary terms in (51)–(53), for the bra-boundary states of the Ising model we have

$$\langle\tilde{I}| \sim 2^{-1}({}_U\langle B(\alpha_{1,1})| + {}_U\langle B(\alpha_{2,3})| + {}_U\langle B(\alpha_{2,1})| + {}_U\langle B(\alpha_{1,3})|) \\ + 2^{-3/4}({}_U\langle B(\alpha_{1,2})| + {}_U\langle B(\alpha_{2,2})|) \tag{95}$$

$$\langle\tilde{\epsilon}| \sim 2^{-1}({}_U\langle B(\alpha_{1,1})| + {}_U\langle B(\alpha_{2,3})| + {}_U\langle B(\alpha_{2,1})| + {}_U\langle B(\alpha_{1,3})|) \\ - 2^{-3/4}({}_U\langle B(\alpha_{1,2})| + {}_U\langle B(\alpha_{2,2})|) \tag{96}$$

$$\langle\tilde{\sigma}| \sim 2^{-1/2}({}_U\langle B(\alpha_{1,1})| + {}_U\langle B(\alpha_{2,3})| - {}_U\langle B(\alpha_{2,1})| - {}_U\langle B(\alpha_{1,3})|). \tag{97}$$

Since these are linear combinations of fixed boundary-charge states ${}_U\langle B(\alpha_{r,s})|$, the correlation functions on the disc with physical boundary conditions $\tilde{I}, \tilde{\epsilon}$ and $\tilde{\sigma}$ are given by linear combinations of fixed boundary-charge correlators (61). Using the global conformal transformation explained in the last section, we shall obtain correlation functions on the half-plane and compare them with existing results.

4.1. One-point functions of spin and energy operators

Let us first consider the spin one-point function. We may choose either $\sigma = \phi_{1,2}$ or $\sigma = \phi_{2,2}$. Let, for definiteness, $(r, s) = (1, 2)$ in the holomorphic part and in order to use the result of the last section, $(\bar{r}, \bar{s}) = (2, 2)$ in the antiholomorphic part. From (70) we immediately note that only the state ${}_U\langle B(\alpha_{1,2})|$ contributes to the one-point function, all other states giving vanishing correlators. For the boundary condition $\tilde{I}$, the one-point function on the disc is

$$\begin{aligned}\langle\tilde{I}|\sigma(z,\bar{z})|0\rangle &= \langle\tilde{I}|V_{1,2}(z)\bar{V}_{2,2}(\bar{z})|0,0;\alpha_0\rangle_U \\ &= 2^{-3/4}{}_U\langle B(\alpha_{1,2})|V_{1,2}(z)\bar{V}_{2,2}(\bar{z})|0,0;\alpha_0\rangle_U \\ &= 2^{-3/4}(1-z\bar{z})^{-1/8}.\end{aligned} \tag{98}$$

Properly normalized one-point function is then,

$$\frac{\langle\tilde{I}|\sigma(z,\bar{z})|0\rangle}{\langle\tilde{I}|0\rangle} = 2^{1/4}(1-z\bar{z})^{-1/8}. \tag{99}$$

This is mapped onto the half-plane by using the conformal transformation (91), as

$$\langle\sigma(w,\bar{w})\rangle_{\tilde{I}} = \langle\sigma(y)\rangle_{\tilde{I}} = 2^{1/4}(2y)^{-1/8} \tag{100}$$

where y is the distance from the boundary. Likewise, boundary spin correlation functions for the conditions $\tilde{\epsilon}$ and $\tilde{\sigma}$ are obtained simply by picking up the coefficients of ${}_U\langle B(\alpha_{1,2})|$ in (96) and (97), and are normalized using $\langle\tilde{\epsilon}|0\rangle = 1/2$ and $\langle\tilde{\sigma}|0\rangle = 1/\sqrt{2}$. On the half-plane, they are

$$\langle\sigma(w,\bar{w})\rangle_{\tilde{\epsilon}} = \langle\sigma(y)\rangle_{\tilde{\epsilon}} = -2^{1/4}(2y)^{-1/8} \tag{101}$$

and

$$\langle\sigma(w,\bar{w})\rangle_{\tilde{\sigma}} = \langle\sigma(y)\rangle_{\tilde{\sigma}} = 0. \tag{102}$$

Hence, $\tilde{I}$ and $\tilde{\epsilon}$ are indeed the fixed (up and down) boundary conditions and $\tilde{\sigma}$ is the free boundary condition, as is stated in [6, 7]. In our Coulomb-gas formalism the relation between one-point functions and the coefficients of Ishibashi states is explained by the neutrality of charges.

Putting, say, $(r, s) = (2, 1)$ and $(\bar{r}, \bar{s}) = (1, 3)$, the energy one-point function is obtained similarly. On the half-plane we have

$$\langle\epsilon(y)\rangle_{\tilde{I}} = \langle\epsilon(y)\rangle_{\tilde{\epsilon}} = (2y)^{-1} \tag{103}$$

$$\langle\epsilon(y)\rangle_{\tilde{\sigma}} = -(2y)^{-1}. \tag{104}$$

4.2. Spin two-point function

Next, let us consider the spin two-point function. Since $\sigma = \phi_{1,2}$, we can use the result of subsection 3.3. There are only two values of boundary charges, $-\alpha_{1,1}$ and $-\alpha_{1,3}$, which give non-trivial contributions to the correlator. The two corresponding states ${}_U\langle B(\alpha_{1,1})|$ and ${}_U\langle B(\alpha_{1,3})|$ give rise to the two conformal blocks I_{I} and I_{II}, respectively, and the correlation function is a linear combination of these conformal blocks with coefficients given by Cardy's states (95)–(97). Then,

$$\begin{aligned}\frac{\langle\tilde{I}|\sigma(z_1,\bar{z}_1)\sigma(z_2,\bar{z}_2)|0\rangle}{\langle\tilde{I}|0\rangle} &= \frac{\langle\tilde{\epsilon}|\sigma(z_1,\bar{z}_1)\sigma(z_2,\bar{z}_2)|0\rangle}{\langle\tilde{\epsilon}|0\rangle} \\ &= I_{\mathrm{I}} + I_{\mathrm{II}}\end{aligned} \tag{105}$$

$$\frac{\langle\tilde{\sigma}|\sigma(z_1,\bar{z}_1)\sigma(z_2,\bar{z}_2)|0\rangle}{\langle\tilde{\sigma}|0\rangle} = I_{\rm I} - I_{\rm II} \tag{106}$$

where the actual forms of $I_{\rm I}$ and $I_{\rm II}$ are given by (83) and (87), with $a = -1/8$, $b = 3/8$ and $\alpha_-^2 = 3/4$. In this case the hypergeometric functions reduce to

$$_2F_1\left(-\frac{1}{4},\frac{1}{4},\frac{1}{2};\eta\right) = \frac{\sqrt{1+\sqrt{1-\eta}}}{\sqrt{2}} \tag{107}$$

$$_2F_1\left(\frac{3}{4},\frac{1}{4},\frac{3}{2};\eta\right) = \frac{\sqrt{2(1-\sqrt{1-\eta})}}{\sqrt{\eta}}. \tag{108}$$

Using (107), (108) and the conformal transformation (88), we find the two-point functions on the half-plane,

$$\begin{aligned}\langle\sigma(w_1,\bar{w}_1)\sigma(w_2,\bar{w}_2)\rangle_{\tilde{I}} &= \langle\sigma(w_1,\bar{w}_1)\sigma(w_2,\bar{w}_2)\rangle_{\tilde{\epsilon}}\\ &= \frac{\tilde{\mathcal{N}}}{\sqrt{2}}\left\{\frac{(w_1-\bar{w}_1)(\bar{w}_2-w_2)}{(w_1-w_2)(\bar{w}_1-\bar{w}_2)(w_1-\bar{w}_2)(\bar{w}_1-w_2)}\right\}^{1/8}\\ &\quad\times\left(\mathcal{N}_{\rm I}\sqrt{\sqrt{1-\eta}+1}+\mathcal{N}_{\rm II}\sqrt{\sqrt{1-\eta}-1}\right)\end{aligned} \tag{109}$$

$$\begin{aligned}\langle\sigma(w_1,\bar{w}_1)\sigma(w_2,\bar{w}_2)\rangle_{\tilde{\sigma}} &= \frac{\tilde{\mathcal{N}}}{\sqrt{2}}\left\{\frac{(w_1-\bar{w}_1)(\bar{w}_2-w_2)}{(w_1-w_2)(\bar{w}_1-\bar{w}_2)(w_1-\bar{w}_2)(\bar{w}_1-w_2)}\right\}^{1/8}\\ &\quad\times\left(\mathcal{N}_{\rm I}\sqrt{\sqrt{1-\eta}+1}-\mathcal{N}_{\rm II}\sqrt{\sqrt{1-\eta}-1}\right)\end{aligned} \tag{110}$$

where $\tilde{\mathcal{N}} = \Gamma(1/4)^2/\Gamma(1/2)$. Studying the behaviours away from the boundary (we accept the convention that two-point functions of bulk operators are normalized as $\langle\phi_i(w_1)\phi_j(w_2)\rangle = \delta_{ij}(w_1-w_2)^{-2h_i}$) and comparing the leading terms with the OPE coefficients of [8], the normalization of the conformal blocks is fixed as $\mathcal{N}_{\rm I} = \mathcal{N}_{\rm II} = \Gamma(1/2)/\Gamma(1/4)^2$.

This result was obtained long time ago [13], by solving a differential equation to find the two conformal blocks $I_{\rm I}$, $I_{\rm II}$, and then fixing the coefficients by considering asymptotic behaviours of the correlation function. The (relative) coefficients of the conformal blocks are now attributed to the coefficients of Cardy's states in our Coulomb-gas approach, although we have used the asymptotic behaviours to fix the normalization of each conformal block.

4.3. Energy two-point function

Finally we derive the energy two-point function in our formalism. As $\epsilon = \phi_{2,1}$, the calculation is parallel to the case of the spin two-point function. From the charge neutrality condition we find that non-vanishing correlators arise from the states ${}_U\langle B(\alpha_{1,1})|$ and ${}_U\langle B(\alpha_{3,1})|$. However, as none of the boundary states (95)–(97) contains ${}_U\langle B(\alpha_{3,1})|$, only ${}_U\langle B(\alpha_{1,1})|$ gives non-trivial contribution to the correlation function. Hence, the energy two-point function does not depend on boundary conditions. After a simple calculation we find, on the half-plane,

$$\begin{aligned}\langle\epsilon(w_1,\bar{w}_1)\epsilon(w_2,\bar{w}_2)\rangle_{\tilde{I},\tilde{\epsilon},\tilde{\sigma}} &= \mathcal{N}\frac{(w_1-\bar{w}_1)(\bar{w}_2-w_2)}{(w_1-w_2)(\bar{w}_1-\bar{w}_2)(w_1-\bar{w}_2)(\bar{w}_1-w_2)}\\ &\quad\times\frac{\Gamma(-1/3)^2}{\Gamma(-2/3)}F(-2,-1/3,-2/3;\eta).\end{aligned} \tag{111}$$

The normalization constant $\mathcal{N}$ is determined as $\mathcal{N} = \Gamma(-2/3)/\Gamma(-1/3)^2$, by considering the off-boundary behaviour. The hypergeometric function turns out to be an algebraic function

$F(-2, -1/3, -2/3; \eta) = 1 - \eta + \eta^2$, and using the coordinates $w_i = x_i + \mathrm{i}y_i$, $\bar{w}_i = w_i^* = x_i - \mathrm{i}y_i (x_i, y_i \in \mathbb{R})$, the correlation function (111) is written as

$$\langle\epsilon(x_1, y_1)\epsilon(x_2, y_2)\rangle_{\tilde{I},\tilde{\epsilon},\tilde{\sigma}} = \frac{4y_1y_2}{[(x_1 - x_2)^2 + (y_1 - y_2)^2][(x_1 - x_2)^2 + (y_1 + y_2)^2]} + \frac{1}{4y_1y_2}. \tag{112}$$

This agrees with the result of [13].

5. Summary

In this paper we have described a novel method of calculating correlation functions of two-dimensional CFT on the disc and on the half-plane. We have used the free-field construction of boundary states developed in [18], and derived boundary correlation functions for the boundary states classified by Cardy's method. The key feature of our formalism is the neutrality of bulk and boundary charges, which associates the coefficients in Cardy's boundary states directly with the linear combinations of conformal blocks. Thus we could unify the two important parts of boundary CFT, namely, boundary correlation functions [13] and consistent boundary states of Cardy [6], by using the Coulomb-gas picture. We have checked the formalism in the Ising model, and shown that our method reproduces the known results.

Cardy's classification of boundary states has been generalized by Lewellen [8] and Pradisi *et al* [9] beyond the diagonal models, and the Coulomb-gas technique is also known to be applicable to more general CFTs, such as WZNW models [29] and CFTs with W-algebra [30–32]. We therefore expect that the method discussed in this paper may be applied to such CFTs relatively easily. In particular, from a string theory point of view, application to WZNW theories seems to be quite fruitful since it would provide an alternative method of finding correlation functions with D-branes on a group manifold. We shall discuss such issues in separate publications [33].

Acknowledgments

I thank Dr John Wheater for useful discussions and constant encouragement. I am also grateful to Professor John Cardy, Dr Matthias Gaberdiel and Dr Ian Kogan for helpful discussions.

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